# Resolving Apparent Paradoxes Between Classical Electromagnetism and Quantum Electrodynamics

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## ABSTRACT

It is widely known within the physics community that the classical energy of electromagnetic waves is amplitude-dependent, while the quantum energy of photons is frequency-dependent. This statement seems to lead to a paradox as the two expressions for energy are dependent on different variables. Furthermore, equating these two expressions leads to the faulty notion that the amplitude and frequency of a wave are interdependent. The resolution to this apparent paradox is found in the Bohr Correspondence Principle which predicts that the classical and quantum theories of light should be consistent when there are numerous photons. Our research gives a formal mathematical method for proving the Bohr Correspondence Principle in this case. We also show how Stefan's Law is obtained from Quantum Electrodynamics in the classical limit that quantum vacuum fluctuations are negligible. Lastly, we show that the effective volume of an individual photon (which is a debated topic in the literature) can be obtained as a temperature-dependent expression using the photon gas model.

## BACKGROUND

### Bohr Correspondence Principle

A physical system on the quantum scale transitions to a system that can be described classically on a macroscopic scale in the limit of large quantum numbers. [1]

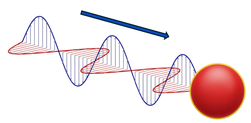


**FIGURE:** A visual illustration relating classical electromagnetic (EM) waves to quantum photons as required by the Bohr Correspondence Principle

### Quantum Electrodynamics (QED)

Quantum Electrodynamics is a theory that describes the interactions between matter and light. It is a quantization of classical electrodynamics. [2]

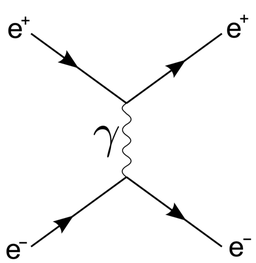



**FIGURE:** A Feynman Diagram for Bhabha Scattering used in QED

### Stefan's Law

The Stefan-Boltzmann law describes the intensity of the thermal radiation emitted by an object as a function of the object's temperature.

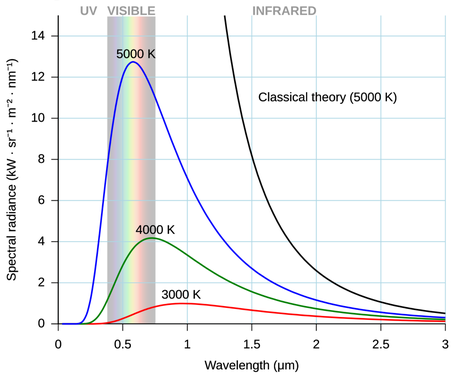



**FIGURE:** The thermal radiation emitted by a black-body has intensity given by Stefan's Law.

## MAIN POINTS

### Paradox of Frequency / Amplitude Dependence of Electromagnetic Intensity

Our analysis begins with the quantized vector potential from Quantum Electrodynamics (QED):

$$\hat{\boldsymbol{A}}(\boldsymbol{r},t) = \sum_{\mathbf{k}} \sum_{p=1}^{2} \sqrt{\frac{\hbar}{2\varepsilon_0 \tilde{V} \omega_{\mathbf{k}}}} \left[ \mathbf{e}_{\mathbf{k},p} \hat{a}_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} - \mathbf{e}^*_{\mathbf{k},p} \hat{a}^\dagger_{\mathbf{k}} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} \right],$$

where $\mathbf{k}$ is the wave vector, $\hat{a}$ and $\hat{a}^\dagger$ are the respective photon annihilation and creation operators, and $\mathbf{e}_p$ represents the two transverse polarization unit vectors of EM waves in vacuum. The electric field is $\boldsymbol{E} = -\partial_t \boldsymbol{A}$ and the magnetic field is $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Also, $\hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} = N_{\mathbf{k}}$, where $N_{\mathbf{k}}$ is the number of photons at a given frequency. Then the intensity is $I_{\mathrm{EM}} = N\hbar c \omega_0 / V$, where $\omega_0$ is the frequency of the given photons, and $V$ is the volume occupied by the photons. This result derived from QED is frequency-dependent which seems to contradict the amplitude-dependent result for the classical EM wave intensity, $I_{\mathrm{EM}} = \frac{1}{2} c \varepsilon_0 E_0^2$, where $E_0$ is the amplitude of the EM wave. [3, 4]

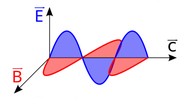



**FIGURE:** EM wave

### Deriving an Effective Photon Volume from the Photon Gas Model

The effective photon volume has been debated in the literature. [5, 6, 7, 8] From the photon gas model given by Wiese [9] and Schwabl [10], the number of photons in a volume $V$ is

$$N = \frac{2\zeta(3)V}{\pi^2} \left( \frac{k_B T}{\hbar c} \right)^3$$

Using $V = N V_{photon}$, we obtain the volume of a photon:

$$V_{photon} = \frac{\pi^2}{2\zeta(3)} \left( \frac{k_B T}{\hbar c} \right)^3$$

The interpretation of this result is that higher temperatures produce more photons. Since the photons are contained by a single fixed volume, the volume available per photon becomes smaller. This resolves the question of the photon volume for the case of a thermodynamic system.

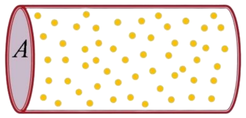



**FIGURE:** Model of photon gas

### Resolving the Quantum / Classical Paradox by Integrating over the EM Spectrum

The paradox is resolved by applying the formal transition from a discrete set of photons to a continuous electromagnetic wave. Starting with the quantized vector potential, $\boldsymbol{A}(\boldsymbol{r},t)$, we arrive at the equation relating the electric field to a sum over photon modes: $\varepsilon_0 \boldsymbol{E}^2 V = \hbar \sum_k \omega_k \left( N_k + \frac{1}{2} \right)$. Instead of considering only a single frequency which leads to the paradox, we take the classical limit by converting this summation into an integral over the photon wavenumber:

$$\sum_k f(k) \to \frac{2\tilde{V}}{(2\pi)^3} \int f(k)(4\pi k^2 dk).$$

Where $\tilde{V}$ is mode volume. Time averaging leads to an intensity given by

$$I_{\mathrm{EM}} = \frac{\hbar}{\pi^2 c^2} \int \left( N_\omega + \frac{1}{2} \right) \omega^3 d\omega .$$

Hence, we find the intensity is not frequency dependent once evaluating the integral over all modes. This can be interpreted as a demonstration of how the BCP resolves the paradox, since numerous photons are required to form a continuous spectrum.

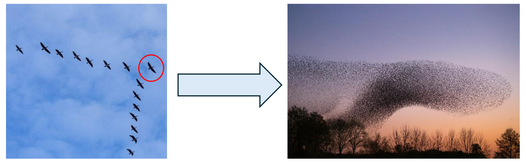


**FIGURE:** Patterns created by birds flying in formation only emerge when there are many birds. This is analogous to how the properties of EM waves only emerge when there are a multitude of photons.

### Recovering the Stefan-Boltzmann Law from QED

Using the Planck photon occupation number, $N(\omega) = \frac{1}{e^{\hbar\omega/(k_B T)} - 1}$, gives $I_{\mathrm{EM}} = 4\sigma T^4 + \frac{\hbar c \omega_p}{2\pi^2 V} \int_0^\infty d\omega$ . The second term is a consequence of the vacuum energy that exists in QED and appears to be divergent. However, the upper bound should be the plasma frequency $\omega_p$ since reflection cannot occur above that frequency. This leads to

$$I_{\mathrm{EM}} = 4\sigma T^4 + \frac{\hbar c \omega_p}{2\pi^2 V}$$

We show this second term is negligible if $V \gg \frac{\hbar c \omega_p}{2\pi^2 4\sigma T^4}$. Using $\omega_p \sim 10^{-16} \frac{rad}{s}$ and $T \sim 1\mathrm{K}$ gives an upper bound of $V \sim 10^{-20} m^3$, which is vastly smaller than any practical cavity size. Hence, we recover the Stefan-Boltzmann Law from QED.

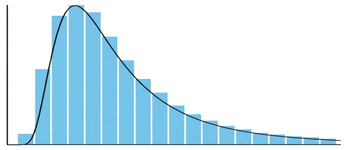
## CONCLUSIONS

### Paradox of Frequency / Amplitude Dependence of Electromagnetic Intensity

Deriving an equation for EM wave intensity from QED gives an expression contradictory to the classical equation for EM wave intensity.

### Resolving the Quantum / Classical Paradox by Integrating over the EM Spectrum

The paradox described above is resolved by converting the summation of discreet modes into an integral over a continuum of frequencies. Time averaging the result leads back to the classical amplitude-dependent intensity.



### Recovering the Stefan-Boltzmann Law from QED

Using the EM wave intensity derived from QED, we showed how Stefan's Law is recovered for a black body cavity of any reasonable size and temperature.

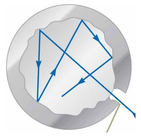


**FIGURE:** Model of a black-body radiation cavity

### Deriving an Effective Photon Volume from the Photon Gas Model

From the work of Wiese [9] and Schwabl [10], we derive an effective photon volume in a photon gas model. It is concluded that the effective photon volume is temperature dependent.

## References

1. Bohr (1918-1923); 2. Sakurai (1967); 3. Atonga, et al. (2024); 4. Goenawan (2021); 5. Xu (2021); 6. Huchin (2021); 7. Liu (2016); 8. Meis (2023); 9. Weise (2010); 10. Schwabl (2000)